交叉研究的感悟

陈木法

(北京师范大学)

北京大学许宝騄讲座

2019 年 3 月 22 日 [redacted]

源于许宝騄先生的一篇论文

北京师范大学现代数学丛书

跳过程与粒子系统

概率核的存在性和转移函数的可微性，北师大学报(1986)

陈木法　著

北京师范大学出版社　1986

源于许宝騄先生的一篇论文

目　录

§1.3 的结果基本上取自许宝騄 [1],他处理了欧氏空间.

许宝騄, 欧氏空间上纯间断的马尔科夫过程的概率转移函数的可微性, 北京大学学报 (1958)

经济模型的最优化：随机矩阵 1988

RANDOM MATRICES

Third Edition

许宝騄,陈家鼎,郑忠国.
随机矩阵的重合性质.
《北大学报》
1979 年 1 期

2004

Madan Lal Mehta

PREFACE TO THE FIRST EDITION

1967 殷涌泉,白志东等

Hsu, P.L.(1939). On the distribution of roots of certain determinantal equations, Ann. Eugenics 9, 250-258.

Though random matrices were first encountered in mathematical statistics by Hsu, Wishart, and others, intensive study of their properties in connection with nuclear physics began with the work of Wigner in the 1950s. Much material has accumulated

百年寿辰

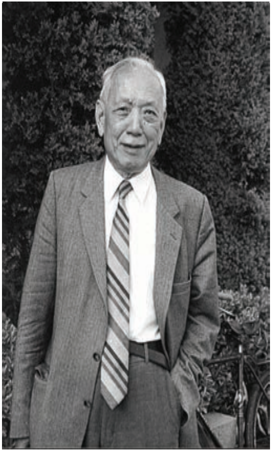
2010 年是華羅庚和許寶騄的百年誕辰。 "数学的进步". 数学通报(2012): 第51卷第9期
數學傳播(2013): 37卷1期

華羅庚

許寶騄

民族英雄

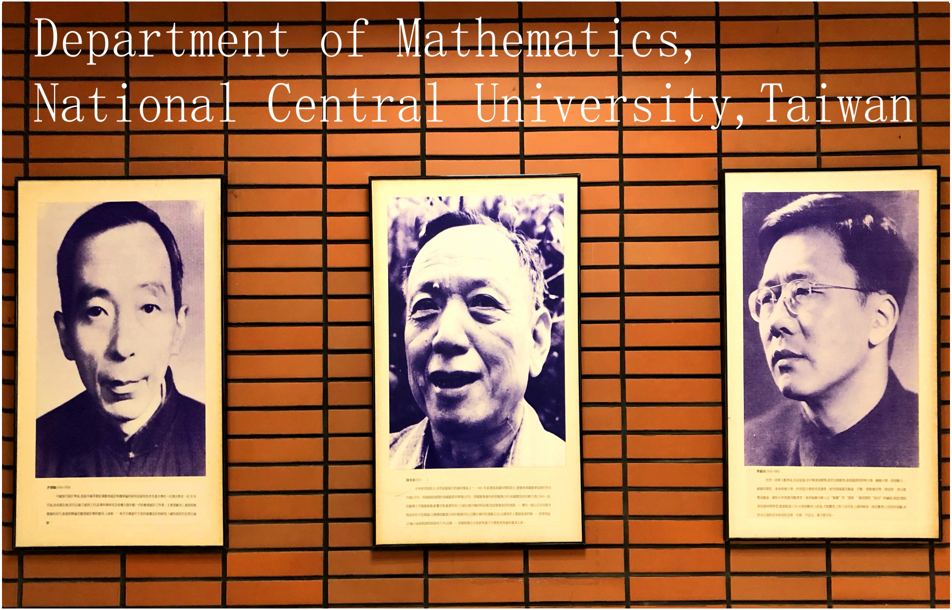
2011 年是陳省身的百年壽辰。華羅庚、許寶騄和陳省身堪稱爲我們的民族英雄。他們幾乎從平地而起,經歷了戰爭等我們無法想像的艱難困苦, 逐步奮鬥成爲頂天立地的第一批華裔數學家。



陳省身

James Simons

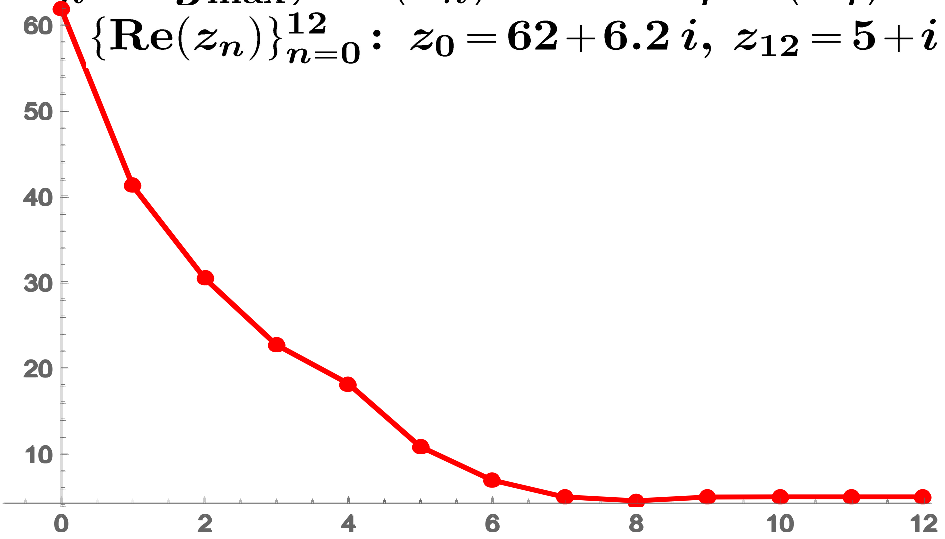
民族英雄


Department of Mathematics,
National Central University, Taiwan

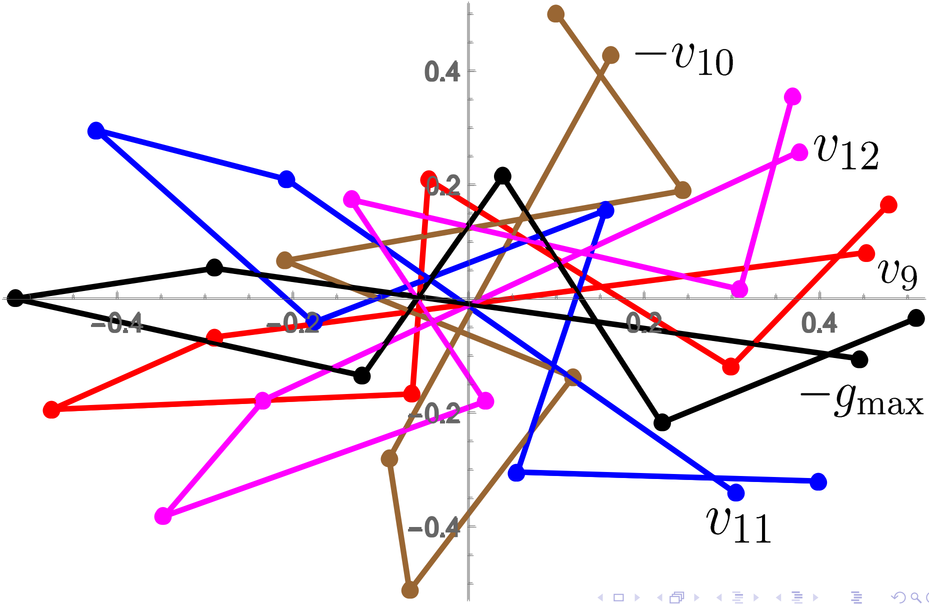
来自计算的挑战: (z_n, v_n). 7 阶复方阵

$$v_n \to g_{\max}, \quad \mathrm{Re}(z_n) \to \max_j \mathrm{Re}(\lambda_j).$$

$$\{\mathrm{Re}(z_n)\}_{n=0}^{12}: \; z_0 = 62 + 6.2\,i, \; z_{12} = 5 + i$$



向量 $\{v_n\}_{n=9}^{12}$ 收敛？ 7 阶复方阵



1. Which matrix has real spectrum?

Real symmetric matrices. $\boldsymbol{E}$: countable.

Real $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ is symmetrizable if

$\exists(\boldsymbol{\mu_k} > \mathbf{0})$ s.t. $\boldsymbol{\mu_i a_{ij}} = \boldsymbol{\mu_j a_{ji}}$. $\boxed{\text{co-zero}}$

$\boldsymbol{A}$: symmetric/selfadjoint on $\boldsymbol{L^2(\mu)}$. $\boldsymbol{\mu}$?

$$\text{Diag}(\boldsymbol{\mu})\boldsymbol{A} = \boldsymbol{A}^*\text{Diag}(\boldsymbol{\mu})$$

$$\Leftrightarrow \text{Diag}(\boldsymbol{\mu})\boldsymbol{A}\,\text{Diag}(\boldsymbol{\mu})^{-1} = \boldsymbol{A}^*$$

$$\Leftrightarrow \text{Diag}(\boldsymbol{\mu})^{1/2}\boldsymbol{A}\text{Diag}(\boldsymbol{\mu})^{-1/2}$$

$\boxed{\text{symmetrizing}}$ $= \text{Diag}(\boldsymbol{\mu})^{-1/2}\boldsymbol{A}^*\text{Diag}(\boldsymbol{\mu})^{1/2}$

对称阵的理论和算法 → 可配称阵

Computational technique. $a_{ij} \geqslant 0,\ i \neq j$

- Invariant/Harmonic meas μ $\boxed{\text{A:irreducible}}$

$$Q := A - \text{DiagMatrix}[A\mathbb{1}] \Rightarrow Q\mathbb{1} = \mathbb{0}.$$

Solve $\mu Q = 0,\ \mu_0 = 1.$ $\boxed{\text{unique}}$

- Quasi-symmetrizing:

$$\hat{A} := \text{Diag}(\mu^{1/2})\, A\, \text{Diag}(\mu^{-1/2}).$$

- A symmetrizable w.r.t μ iff $\hat{A}$ symmetric: $\hat{A}^* - \hat{A} = 0?$ $\boxed{\text{Diagonals, summing}}$

Criterion and representation. $a_{ij} \geqslant 0,\ i \neq j$

Path: $i_0 \to i_1 \to \cdots \to i_n,\ a_{i_k i_{k+1}} > 0$

$$\mu_{i_0} \frac{a_{i_0 i_1}}{a_{i_1 i_0}} = \mu_{i_1}$$

$$\boxed{a_{i_k i_{k+1}} \neq 0,\ \frac{a_{i_0 i_1}}{a_{i_1 i_0}} > 0}$$

$$\mu_{i_0} \frac{a_{i_0 i_1}}{a_{i_1 i_0}} \cdot \frac{a_{i_1 i_2}}{a_{i_2 i_1}} = \mu_{i_1} \frac{a_{i_1 i_2}}{a_{i_2 i_1}} = \mu_{i_2}$$

$$\frac{a_{i_0 i_1}}{a_{i_1 i_0}} \frac{a_{i_1 i_2}}{a_{i_2 i_1}} \cdots \frac{a_{i_{n-1} i_n}}{a_{i_n i_{n-1}}} = \frac{\mu_{i_n}}{\mu_{i_0}} \qquad (\star)$$

$i_n = i_0$: Kolmogorov (1936) circle theorem.

$\mu_{i_0} = 1$, path-indep, field theory, smallest

Hermitizable/complex symmetrizable?[a_{ii} 实

$\boldsymbol{A} = (\boldsymbol{a}_{ij})$ is Hermitizable [可厄米] if

$\exists(\boldsymbol{\mu}_k > \mathbf{0})$ s.t. $\boldsymbol{\mu}_i \boldsymbol{a}_{ij} = \boldsymbol{\mu}_j \bar{\boldsymbol{a}}_{ji}$. co-zero

$\boldsymbol{A}$: symmetric/selfadjoint on complex $\boldsymbol{L}^2(\boldsymbol{\mu})$

$$\text{Diag}(\boldsymbol{\mu})\boldsymbol{A} = \boldsymbol{A}^{\boldsymbol{H}}\text{Diag}(\boldsymbol{\mu}) \quad \boxed{\boldsymbol{A}^{\boldsymbol{H}} := (\bar{\boldsymbol{A}})^*}$$

$$\Leftrightarrow \text{Diag}(\boldsymbol{\mu})\boldsymbol{A}\,\text{Diag}(\boldsymbol{\mu})^{-1} = \boldsymbol{A}^{\boldsymbol{H}}$$

$$\Leftrightarrow \text{Diag}(\boldsymbol{\mu})^{1/2}\boldsymbol{A}\text{Diag}(\boldsymbol{\mu})^{-1/2} \quad \boxed{\text{Hermitizing}}$$

$$= \text{Diag}(\boldsymbol{\mu})^{-1/2}\boldsymbol{A}^{\boldsymbol{H}}\text{Diag}(\boldsymbol{\mu})^{1/2}.$$

厄米阵的理论和算法 → 可厄米阵

Criterion for Hermitizability

$a_{ij} \geqslant 0, \ i \neq j$. A.N. Kolmogorov (1936), ..., Z.T. Hou & C. (1979).

Theorem (C. (2018))

Complex $A = (a_{ij})$ is Hermitizable iff two conditions hold simultaneously.

- For each pair i, j, either $a_{ij} \& a_{ji} = 0$ or $a_{ij} a_{ji} > 0$ $(\Leftrightarrow a_{ij}/\bar{a}_{ji} > 0)$.
- The circle condition holds for each smallest closed path without round-trip.

Tridiagonal/Birth-Death matrix $A \sim (a_k, -c_k, b_k)$, $E = \{k \in \mathbb{Z}_+ : 0 \leqslant k < N+1\}$

$$Q = \begin{pmatrix} -c_0 & b_0 & & & 0 \\ a_1 & -c_1 & b_1 & & \\ & a_2 & -c_2 & b_2 & \\ & & \ddots & \ddots & b_{N-1} \\ 0 & & & a_N & -c_N \end{pmatrix},$$

$(a_k), (b_k), (c_k)$: complex sequences.

BD: $a_k > 0$, $b_k > 0$, $c_k = a_k + b_k$, $c_N \geqslant a_N$.

Tridiagonal matrix: $T \sim (a_k, -c_k, b_k)$

Theorem

The tridiagonal T is Hermitizable iff the following two conditions hold simultaneously.

- The diagonals (c_k) are real.
- Either $a_{i+1} \& b_i = 0$ or $a_{i+1} b_i > 0$

分块 $(\Leftrightarrow b_i/\bar{a}_{i+1} > 0)$.

Then $$\mu_0 = 1, \ \mu_k = \mu_{k-1} \frac{b_{k-1}}{\bar{a}_k}.$$

可逆马尔可夫过程

KENI MAERKEFU GUOCHENG

作者（按姓氏笔划为序）

陈木法 汪培庄 侯振挺 郭青峰

钱 敏 钱敏平 龚光鲁

湖南科学技术出版社

1979·长沙

pp. 194--142

47 pages

"From Markov Chains to Nonequilibrium Particle Systems", World Scientific 1992, 2004

Chapers 7 & 11; Section 14.3

A family of \mu

Theorem/Algorithm. Let $T\sim(a_k, -c_k, b_k)$.

Supp $m := \sup_k(-c_k+|a_k|+|b_k|)^+ < \infty$ and set $u_k = a_k b_{k-1}$, $\tilde c_k = m + c_k$, $0 \leqslant k < N+1$. Next, define $\tilde b_0 = \tilde c_0 > 0$,

$$\begin{cases} \tilde b_k = \tilde c_k - u_k/\,\tilde b_{k-1}, \;\; \tilde a_k = \tilde c_k - \tilde b_k, \\ \qquad 1 \leqslant k < N \\ \tilde a_N = u_N/\,\tilde b_{N-1} \;\; \text{if } N < \infty. \end{cases}$$

Then $\tilde a_k$, $\tilde b_k > 0$; $\tilde c_N \geqslant \tilde a_N$ if $N < \infty$. Moreover, [Hermitizable] T is isospectral to $\widetilde Q \sim (\tilde a_k, -\tilde c_k, \tilde b_k)$.

Explicit $u_k = a_k b_{k-1} = |a_k b_{k-1}|$ & $\tilde{c}_k$

$$\tilde{b}_k = \tilde{c}_k - \cfrac{u_k}{\tilde{c}_{k-1} - \cfrac{u_{k-1}}{\tilde{c}_{k-2} - \cfrac{u_{k-2}}{\ddots\ \tilde{c}_2 - \cfrac{u_2}{\tilde{c}_1 - \cfrac{u_1}{\tilde{c}_0}}}}}$$

$$\tilde{a}_k = \tilde{c}_k - \tilde{b}_k,\ \ k < N; \qquad \tilde{a}_N = u_N \big/ \tilde{b}_{N-1}.$$

Isometry and isospectrality

Every irreducible Hermitizable tridiagonal matrix T on complex $L^2(\mu)$ is isospectral to a birth–death [BD] Q-matrix $\widetilde{Q}$ on real $L^2(\tilde{\mu})$: $\tilde{\mu} = |h|^2\mu$, $Th = 0$ on $[0, N)$.

$\boxed{\widetilde{Q}=\text{Diag}(h)^{-1}T\,\text{Diag}(h)}$

Explicit h

$$h_0 = 1,\ h_n = h_{n-1}\frac{\tilde{b}_{n-1}}{b_{n-1}},\ 1 \leqslant n \leqslant N.$$ $\boxed{\text{超}}$

Isometry: $f \to \tilde{f} := f/h$, $\|f\|_\mu = \|\tilde{f}\|_{\tilde{\mu}}$

Isospectrality: $(Tf, f)_\mu = \left(\widetilde{Q}\tilde{f}, \tilde{f}\right)_{\tilde{\mu}}$

Mathematical methods

• h-transform [C. & Xu Zhang, 2014]:

$L = \Delta + V \to \widetilde{L} = \Delta + \tilde{b}^h \nabla, \quad Lh = 0,$

L on $L^2(\mathrm{d}x)$ is isospectral to $\widetilde{L}$ on

$L^2(\tilde{\mu}) := L^2(|h|^2 \mathrm{d}x)$. 做计算

Criteria for discrete spectrum (one dim)

• Feynman-Kac semigroup: real $\boxed{L = \Delta + V}$

$$T_t f(x) = \mathbb{E}_x \left\{ \exp\left[\int_0^t V(w_s) \mathrm{d}s \right] f(w_t) \right\}.$$

Remove "tridiagonal" condition

Hermitizable $\Leftrightarrow$ $\text{Diag}(\boldsymbol{\mu})^{1/2} \boldsymbol{A} \text{Diag}(\boldsymbol{\mu})^{-1/2}$ is Hermitian

Hermite $\xrightarrow{\boldsymbol{U}}$ tridiagonal, real, symmetric

$\rightarrow$ BD (by our result)

Householder transformation

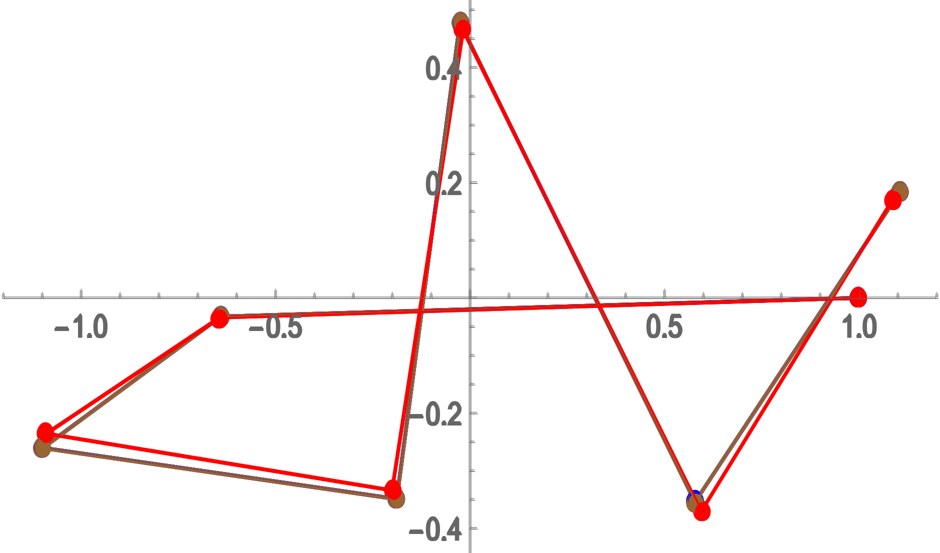
Top 10 in 20th century $\boxed{\text{C.G.J. Jacobi, 1846}}$

Results from BD $\rightarrow$ Hermitizable. $\boxed{\text{173 years}}$

Hermite$\rightarrow$Hermitizable: 均匀$\rightarrow$非均匀介质. BD.

自恰的理论终有其客观的存在.

Hermitian$\rightarrow$Non-Hermitian, $\mathscr{P}\mathscr{T}$-symmetry



$\tilde{v}_9$: red. $\tilde{v}_{10}$, $\tilde{v}_{11}$, $\tilde{v}_{12}$ and $\tilde{g}$ are coincided

2. Computing the maximal eigenpair. Trid

$$E = \{k \in \mathbb{Z}_+ : 0 \leqslant k < N+1\}.$$

$$Q = \begin{pmatrix} -3 & 2 & & & 0 \\ 1 & -3 & 2 & & \\ & 1 & -3 & 2 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & 1 & -3 \end{pmatrix},$$

$$\lambda_0 + 3 = 2\sqrt{2}\cos\frac{2\pi}{N+2}$$

$$g_0(j) = 2^{-(j+1)/2}\sin\frac{(j+1)\pi}{N+2}, \quad j \in E.$$

Shifted Inverse Algorithm

Given initial $(v_0, z_0) \approx (g_0, \lambda_0(-Q))$, suppose for $k \geqslant 1$, we have (v_{k-1}, z_{k-1}). Let w_k solve the equation

$$\big(-Q - z_{k-1}I\big)w_k = v_{k-1}.$$

Next, let $v_k = w_k/\|w_k\|, \qquad z_k = \delta_k^{-1}$

Then $(v_k, z_k) \to \big(g_0, \lambda_0(-Q)\big)$, $k \to \infty$.

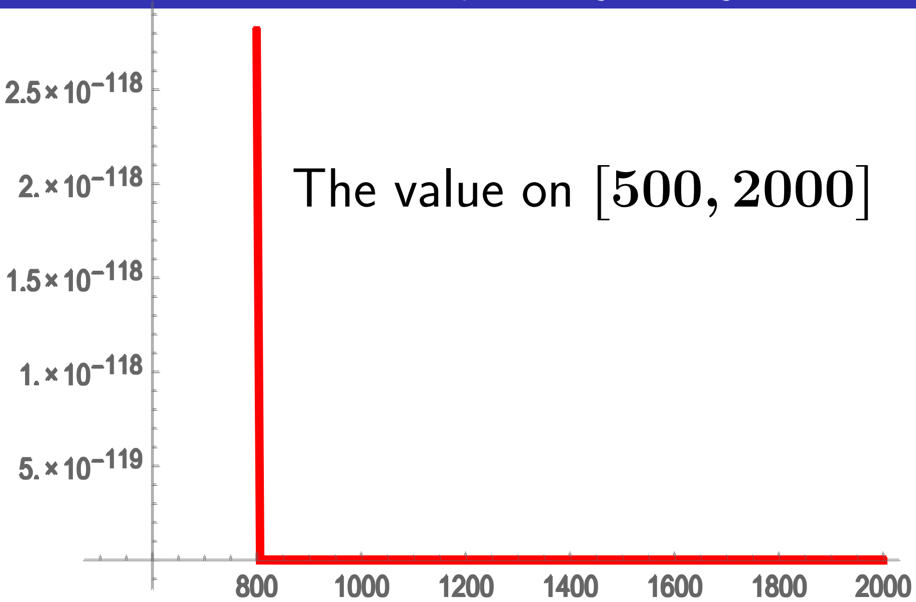
Isospectral matrix having conservative rows

$$E = \{k \in \mathbb{Z}_+ : 0 \leqslant k < N + 1\}.$$

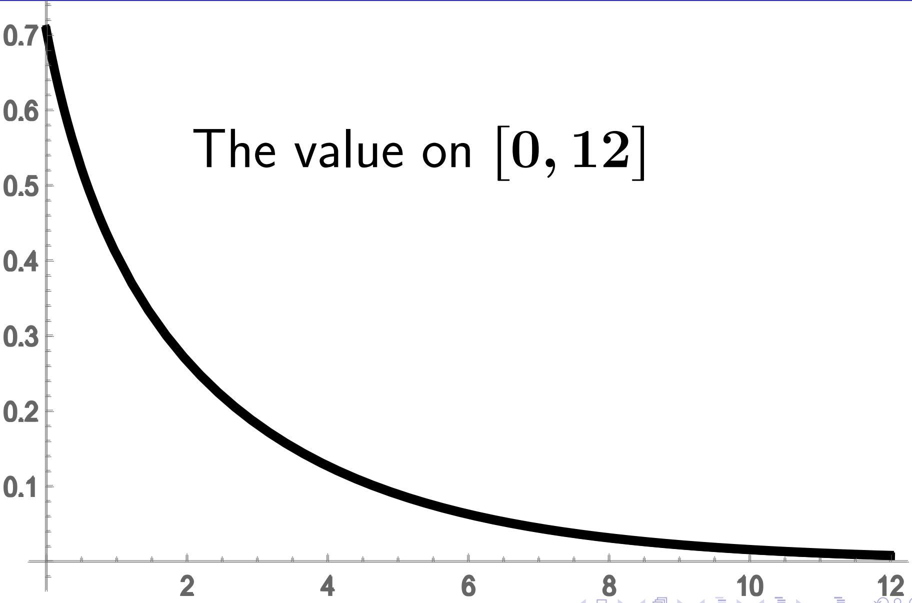
$$\tilde{Q} = \begin{pmatrix} -3 & \frac{2^2-1}{2-1} & & & 0 \\ \frac{2^2-2}{2^2-1} & -3 & \frac{2^3-1}{2^2-1} & & \\ & \frac{2^3-2}{2^3-1} & -3 & \frac{2^4-1}{2^3-1} & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \frac{2^{N+1}-2}{2^{N+1}-1} & -3 \end{pmatrix},$$

$$\lambda_0 + 3 = 2\sqrt{2}\cos\frac{2\pi}{N+2} \approx 2.82843, \ N \geqslant 2562.$$

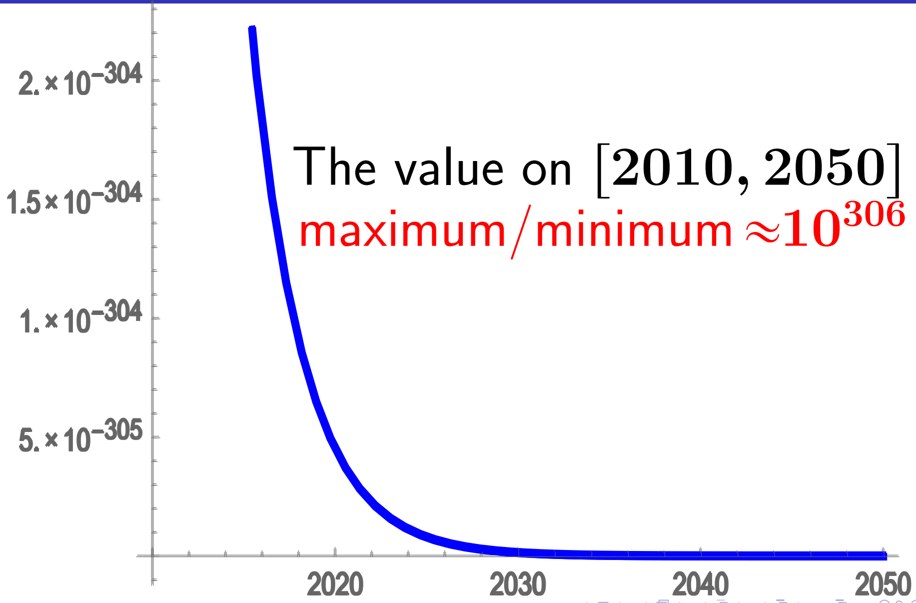
The initial vector for $\tilde{Q}$ decays very fast



The initial vector for $\tilde{Q}$ decays very fast $\mathbf{0.7}\downarrow$



The initial vector for $\tilde{Q}$ decays very fast **0.7** $\downarrow$



The value on $[2010, 2050]$

maximum/minimum $\approx 10^{306}$

Q^{Sym}: symmetrization of $\widetilde{Q}$

$$Q^{\text{Sym}} = \begin{pmatrix} -3 & \sqrt{2} & & & 0 \\ \sqrt{2} & -3 & \sqrt{2} & & \\ & \sqrt{2} & -3 & \sqrt{2} & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \sqrt{2} & -3 \end{pmatrix},$$

For the symmetrized matrix,

maximum/minimum $= \sqrt{2} \ll 10^{306}$.

Sum of each row $\neq 0$, loss of martial arts.

Isospectral transform $\Rightarrow$ non-symmetry.

Unsolvable circulation. 处于绝境

奇思妙想，绝地逢生

$N+1$	$z^{(0)}$	$z^{(1)}$	$z^{(2)}$	$z^{(3)}$
10^2	.171573	.172686	.172934	.172941
5×10^2	.171573	.171628	.171618	
10^3	.171573	.171584	.171587	
5×10^3	.171573	.171573	(1.597s)	
10^4	.171573	.171573	(6.578s)	
1.5×10^4	.171573	.171573	(29.16s)	

C.& Y.S. Li

Tridiagonal Q

2016 h-transformation (2014)

Hermitizable T, A

2018 / 2014

$\tilde{Q}$. Initial $(v^{(0)}, z^{(0)})$

2016

Basic esti (2010)

2016

2018

Coupling of $\tilde{Q}$ $\& Q^{\text{sym}}$ (1991)

2018

Iterative equation
$$\left(-Q^{\text{sym}} - z^{(k-1)} I\right) \times w^{(k)} = v^{(k-1)}$$

$z^{(k)} = \zeta_k^{-1}$ (2010)

2018

2016 Default algorithm

2018

L.H. Thomas algo (1949)

shift $z^{(k)} = v^{(k)*}(-\tilde{Q})v^{(k)}$

2017

$z^{(k)} = \delta_k^{-1}$ (2010)

3. Improved global algorithm[2017]

General nonnegative matrix.

Example.

$$A = (a_{kj})_{N\times N}, \quad a_{kj} = \frac{2^{k\wedge j} - 1}{2^j} + \mathbb{1}_{\{j\leqslant k\}}.$$

Using Mathematica (version 11.3),
Maximal eigenvector is computable iff $N \leqslant 39$. Without term $\mathbb{1}_{\{j\leqslant k\}}$, software is available iff $N \leqslant 11$.

Quasi-symmetrization.

Quasi-symmetrizing:

$$\widehat{A} := \mathrm{Diag}(\mu^{1/2})A\,\mathrm{Diag}(\mu^{-1/2}).$$

$\mu Q = 0$? Maybe not practical!

$$a_{kj} = \frac{2^{k\wedge j} - 1}{2^j} + \mathbb{1}_{\{j \leqslant k\}}. \qquad \mu_k = 2^{-k}.$$

$$\mu_k = \left(2 \times 10^{2/39}\right)^{-k}.$$

Results of quasi-symmetrization

$\mu_k = \left(2 \times 10^{2/39}\right)^{-k}$. $\qquad 1650 \gg 39$

N	523	800	1000	1500	1650
$z^{(1)}$	8.96625 8.99793	8.97632 8.99911	8.98126 8.99943	8.98993 8.99975	8.99086 8.99979
$z^{(2)}$	8.99734 8.99746	8.99885 8.99891	8.99926 8.9993	8.99967 8.99969	8.99973 8.99974
$z^{(3)}$	8.99744 8.99744	8.9989 8.9989	8.9993 8.9993	8.99969 8.99969	8.99974 8.99974

Here $z^{(k)}$ denotes the output of Step k of the lower/upper estimates of $\rho(A)$.

Mathematica

The model comes from single death $Q = (q_{ij} : i, j = 1, \ldots, N)$

$A = (a_{kj}) := (-Q)^{-1}$: Not co-zero!

$$Q = \begin{pmatrix} - & * & & & & \\ + & - & * & & \text{\huge *} & \\ & + & - & * & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & * \\ & & & & + & - \end{pmatrix}.$$

$A = (-Q)^{-1} > 0$

Lemma

Let Q: a Q-matrix (not necessarily conservative) on E, derived by $\{P_t\}_{t\geqslant 0}$. Then, $(-Q)^{-1}$ exists and nonnegative (pointwise, may be ∞). If moreover, Q is irreducible and $\{P_t\}_{t\geqslant 0}$ is transient, then $(-Q)^{-1}$ is finite and positive.

Proof As an increasing limit of $\boxed{\text{weak}}$

$$(\lambda I - Q)^{-1} = \int_0^\infty e^{-\lambda t} P_t \, \mathrm{d}t, \qquad \lambda \downarrow 0.$$

概率与物理、数学其它分支的交叉

- 非平衡统计物理数学基础: 反应扩散过程较系统理论
- 相变数学工具: 各种稳定性的速度估计(4 卷本)

基础: 宏伟; 随机: 细腻; 物理: 直觉; 分析: 精巧; 代数: 高雅; 计算: 大气; 应用: 迷人

感悟 科研的 4 种境界, 学科交叉渗透

华罗庚科普著作选集, 第二部分末文

- 研究要攻得进去，还要打得出来.
- 鉴别一个学问家，要看广度和深度.
  单是深，可成为不错的专家，但对整个科学的发展不足道.
  单是广, 可欺外行, 难有实质性成就.
- 学科之间实际上无不可逾越的鸿沟.
- 做学问需要有绅士风度, 新思想, 打个洞. 根据地, "漫"而非跳.

新奇想法，超，贵人相助，感谢老师

感恩之语

胡迪鹤、

钱敏平、龚光鲁、钱敏、

[+ Martin L.Silverstein, 1979.

Masatoshi Fukushima-2014]

陈家鼎、江泽培、

陈大岳, 1990's, 讨论班,

重点项目, 97–98 中、俄合作项目, 2002

张恭庆、姜伯驹、

耿直、文兰、丁伟岳......

http://math0.bnu.edu.cn/~chenmf

The end!

Thank you, everybody!

谢谢大家!

43