交叉研究的感悟

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第三届江苏师范大学概率统计青年学者会议

2019 年 10 月 12 日

交叉研究概述

1978–1992 (15 years): 概率+统计物理.
跳过程与反应扩散过程
From Markov Chains to... (1992, 2004)

1988– (30 years): ∞ 维数学工具 $\rightarrow$ 相变
$\leftrightarrow$ 谱理论, 几何, 调和分析...基础数学
Eigenvalues, Inequalities, and Ergodic Theory

2015– (4 years): 计算数学

相变、经济优化

1972–1978: 优选法. 运筹学
1988–1991: 经济最优化. 应用数学

交叉研究概述

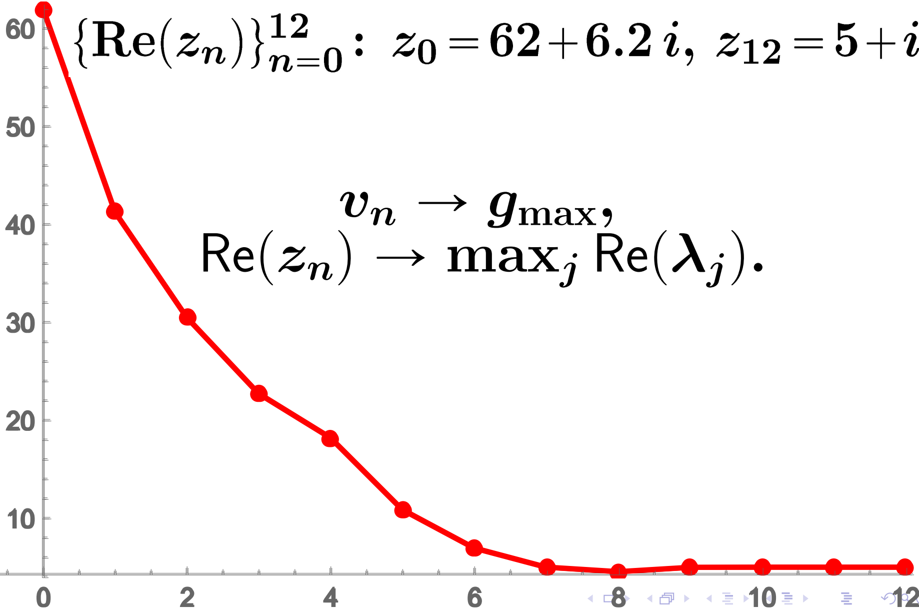
主页: http://math0.bnu.edu.cn/~chenmf

四篇贺寿文: 严士健、王梓坤 90 寿辰
侯振挺、杨向群 80 寿辰 (共 48 页)

英文专著栏目.
四卷本文集 (1993–): vol 1 – vol 4

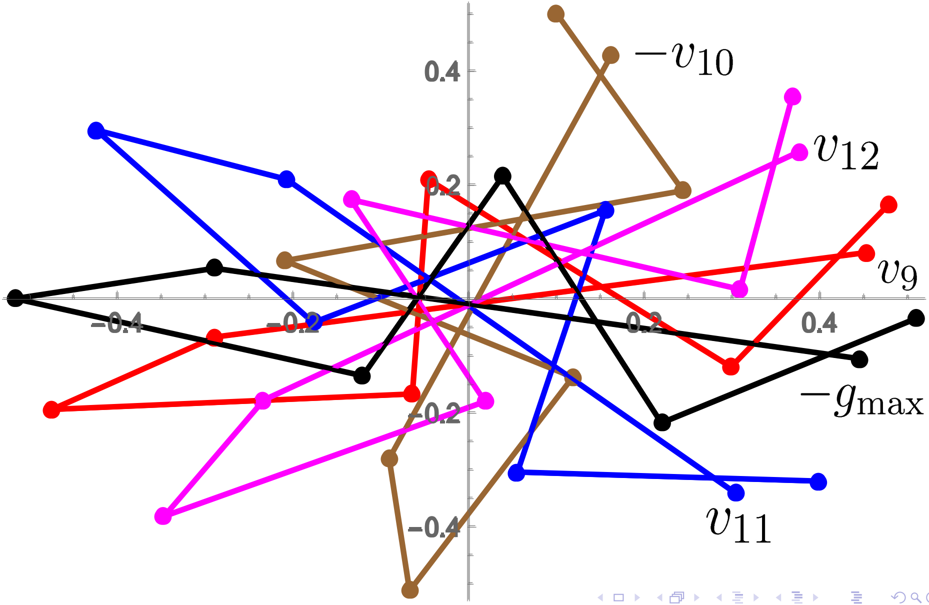
来自计算的挑战: (z_n, v_n). 7 阶复方阵

$\{\mathrm{Re}(z_n)\}_{n=0}^{12}$: $z_0 = 62 + 6.2\,i$, $z_{12} = 5 + i$



$$v_n \to g_{\max},$$
$$\mathrm{Re}(z_n) \to \max_j \mathrm{Re}(\lambda_j).$$

向量 $\{v_n\}_{n=9}^{12}$ 收敛? 7 阶复方阵



1. Symmetrizable/Hermitizable matrix

Real symmetric matrices. $\boldsymbol{E}$: countable.

Real $\boldsymbol{A} = (\boldsymbol{a_{ij}})$ is symmetrizable if

$\exists(\boldsymbol{\mu_k} > \mathbf{0})$ s.t. $\boldsymbol{\mu_i a_{ij}} = \boldsymbol{\mu_j a_{ji}}$. $\boxed{\text{co-zero}}$

$\boldsymbol{A}$: symmetric/selfadjoint on $\boldsymbol{L^2(\mu)}$. $\boldsymbol{\mu}$?

$$\text{Diag}(\boldsymbol{\mu})\boldsymbol{A} = \boldsymbol{A}^*\text{Diag}(\boldsymbol{\mu})$$

$$\Leftrightarrow \text{Diag}(\boldsymbol{\mu})\boldsymbol{A}\,\text{Diag}(\boldsymbol{\mu})^{-1} = \boldsymbol{A}^*$$

$$\Leftrightarrow \text{Diag}(\boldsymbol{\mu})^{1/2}\boldsymbol{A}\text{Diag}(\boldsymbol{\mu})^{-1/2}$$

$\boxed{\text{symmetrizing}}$ $= \text{Diag}(\boldsymbol{\mu})^{-1/2}\boldsymbol{A}^*\text{Diag}(\boldsymbol{\mu})^{1/2}$

对称阵的理论和算法 → 可配称阵

Criterion and representation. $a_{ij} \geqslant 0,\ i \neq j$

Path: $i_0 \to i_1 \to \cdots \to i_n,\ a_{i_k i_{k+1}} > 0$

$$\mu_{i_0} \frac{a_{i_0 i_1}}{a_{i_1 i_0}} = \mu_{i_1}$$

$$\boxed{a_{i_k i_{k+1}} \neq 0,\ \frac{a_{i_0 i_1}}{a_{i_1 i_0}} > 0}$$

$$\mu_{i_0} \frac{a_{i_0 i_1}}{a_{i_1 i_0}} \cdot \frac{a_{i_1 i_2}}{a_{i_2 i_1}} = \mu_{i_1} \frac{a_{i_1 i_2}}{a_{i_2 i_1}} = \mu_{i_2}$$

$$\frac{a_{i_0 i_1}}{a_{i_1 i_0}} \frac{a_{i_1 i_2}}{a_{i_2 i_1}} \cdots \frac{a_{i_{n-1} i_n}}{a_{i_n i_{n-1}}} = \frac{\mu_{i_n}}{\mu_{i_0}} \qquad (\star)$$

$i_n = i_0$: Kolmogorov (1936) circle theorem.

$\mu_{i_0} = 1$, path-indep, field theory, smallest

$\boldsymbol{A} = (\boldsymbol{a}_{ij})$ is Hermitizable [可厄米] if

$\exists(\boldsymbol{\mu}_k > \mathbf{0})$ s.t. $\boldsymbol{\mu}_i \boldsymbol{a}_{ij} = \boldsymbol{\mu}_j \bar{\boldsymbol{a}}_{ji}$. co-zero

$\boldsymbol{A}$: symmetric/selfadjoint on complex $\boldsymbol{L}^2(\boldsymbol{\mu})$

$$\text{Diag}(\boldsymbol{\mu})\boldsymbol{A} = \boldsymbol{A}^{\boldsymbol{H}}\text{Diag}(\boldsymbol{\mu}) \quad \boxed{\boldsymbol{A}^{\boldsymbol{H}} := (\bar{\boldsymbol{A}})^*}$$

$$\Leftrightarrow \text{Diag}(\boldsymbol{\mu})\boldsymbol{A}\,\text{Diag}(\boldsymbol{\mu})^{-1} = \boldsymbol{A}^{\boldsymbol{H}}$$

$$\Leftrightarrow \text{Diag}(\boldsymbol{\mu})^{1/2}\boldsymbol{A}\text{Diag}(\boldsymbol{\mu})^{-1/2} \quad \boxed{\text{Hermitizing}}$$

$$= \text{Diag}(\boldsymbol{\mu})^{-1/2}\boldsymbol{A}^{\boldsymbol{H}}\text{Diag}(\boldsymbol{\mu})^{1/2}.$$

厄米阵的理论和算法 → 可厄米阵

Criterion for Hermitizability

$a_{ij} \geqslant 0,\ \ i \neq j$. A.N. Kolmogorov (1936), ..., Z.T. Hou & C. (1979).

Theorem (C. (2018))

Complex $A = (a_{ij})$ is Hermitizable iff two conditions hold simultaneously.

- For each pair i, j, either $a_{ij} \& a_{ji} = 0$ or $a_{ij} a_{ji} > 0$ ($\Leftrightarrow a_{ij}/\bar{a}_{ji} > 0$).
- The circle condition holds for each smallest closed path without round-trip.

Tridiagonal/Birth-death matrix

$T \sim (a_k, -c_k, b_k), \quad E = \{k \in \mathbb{Z}_+ : 0 \leqslant k < N+1\}$

$$\begin{matrix} T \\ Q \end{matrix} = \begin{pmatrix} -c_0 & b_0 & & & 0 \\ a_1 & -c_1 & b_1 & & \\ & a_2 & -c_2 & b_2 & \\ & & \ddots & \ddots & b_{N-1} \\ 0 & & & a_N & -c_N \end{pmatrix},$$

$(a_k), (b_k), (c_k)$: complex sequences.

BD: $a_k > 0$, $b_k > 0$, $c_k = a_k + b_k$, $c_N \geqslant a_N$.

Tridiagonal matrix: $T \sim (a_k, -c_k, b_k)$

Theorem

The tridiagonal T is Hermitizable iff the following two conditions hold simultaneously.

- The diagonals (c_k) are real.
- Either $a_{i+1} \& b_i = 0$ or $a_{i+1} b_i > 0$

分块 $\quad (\Leftrightarrow b_i / \bar{a}_{i+1} > 0)$.

Then $$\mu_0 = 1, \ \mu_k = \mu_{k-1} \frac{b_{k-1}}{\bar{a}_k}.$$

可逆马尔可夫过程

KENI MAERKEFU GUOCHENG

作者(按姓氏笔划为序)

陈木法 汪培庄 侯振挺 郭青峰

钱　敏 钱敏平 龚光鲁

湖南科学技术出版社

1979·长沙

pp. 194--242

47 pages

From Markov Chains to Nonequilibruim Particle Systems, World Sci. (1992^{1st}, 2004^{2nd}).

Chapter 7: finite prod space. With Hou.

Chapter 11: ∞ prod space, equilibruim.

§14.3: ∞ prod space, nonequilibruim.

With Shi-Jian Yan et al. $\boxed{\{-1,1\}^{\mathbb{Z}^d},\ \mathbb{Z}_+^{\mathbb{Z}^d}}$

∞ many closed path. Single condition.

Allow a family of μ [phase transition].

Theorem/Algorithm

Given Hermitizable $T \sim (a_k, -c_k, b_k)$ with $c_k \geqslant |a_k| + |b_k|$ (or $\tilde{c}_k = c_k + m$) Then $\exists$ an explicit birth–death matrix $\widetilde{Q} \sim (\tilde{a}_k, -\tilde{c}_k, \tilde{b}_k)$ such that T is isospectral to $\widetilde{Q}$.

In general, we have $\tilde{c}_N \geqslant \tilde{a}_N$. We assume that $\tilde{c}_N > \tilde{a}_N$ in what follows. The case $\tilde{c}_N = \tilde{a}_N$ was also treated in the published paper (2018, 2019).

$\boxed{\text{Explicit } u_k := a_k b_{k-1} > 0 \ \& \ c_k =: \tilde{c}_k}$

$$\tilde{b}_k = c_k - \cfrac{u_k}{c_{k-1} - \cfrac{u_{k-1}}{c_{k-2} - \cfrac{u_{k-2}}{\ddots\, c_2 - \cfrac{u_2}{c_1 - \cfrac{u_1}{c_0}}}}}$$

$$\boxed{\tilde{b}_k = c_k - u_k/\tilde{b}_{k-1}, \ \tilde{b}_0 = c_0}$$

$$\tilde{a}_k = c_k - \tilde{b}_k, \ k < N; \quad \tilde{a}_N = u_N/\tilde{b}_{N-1}.$$

Theorem

Every irreducible Hermitizable tridiagonal matrix T on complex $L^2(\mu)$ is isospectral to a birth–death [BD] Q-matrix $\widetilde{Q}$ on real $L^2(\tilde{\mu})$: $\tilde{\mu} = |h|^2\mu$, $Th = 0$ on $[0, N)$.

Explicit h

$$h_0 = 1,\ h_n = h_{n-1}\frac{\tilde{b}_{n-1}}{b_{n-1}}, \qquad 1 \leqslant n \leqslant N.$$

$$\widetilde{Q} = \mathrm{Diag}(h)^{-1} T\, \mathrm{Diag}(h).$$

General solution, not solvable!

Particular one. Four years

Mathematical methods

• h-transform [C. & Xu Zhang, 2014]:

$L = \Delta + V \to \widetilde{L} = \Delta + \tilde{b}^h \nabla$, $Lh = 0$,

L on $L^2(\mathrm{d}x)$ is isospectral to $\widetilde{L}$ on

$L^2(\tilde{\mu}) := L^2(|h|^2 \mathrm{d}x)$. 做计算

Criteria for discrete spectrum (one dim)

• Feynman-Kac semigroup: real $\boxed{L = \Delta + V}$

$$T_t f(x) = \mathbb{E}_x \left\{ \exp\left[\int_0^t V(w_s) \mathrm{d}s \right] f(w_t) \right\}.$$

Remove "tridiagonal" condition

Hermitizable $\Leftrightarrow \mathrm{Diag}(\boldsymbol{\mu})^{1/2} \boldsymbol{A} \mathrm{Diag}(\boldsymbol{\mu})^{-1/2}$ is Hermitian

Hermite $\xrightarrow{\boldsymbol{U}}$ tridiagonal, real, symmetric

$\rightarrow$ BD (by our result) [Blocks]

Householder transformation

Top 10 in 20th century $\boxed{\text{C.G.J. Jacobi, 1846}}$

Results from BD $\rightarrow$ Hermitizable. $\boxed{\text{173 years}}$

Hermite$\rightarrow$Hermitizable: 均匀$\rightarrow$非均匀介质. BD.

自恰的理论终有其客观的存在.

Hermitian$\rightarrow$Non-Hermitian, $\mathscr{P}\mathscr{T}$-symmetry

Quantum mechanics

Two characters of quantum mechanics: complex operator (wave) and

real spectrum (observation)

Schrödinger eq.: $i\hbar\dot{\psi}(t) = (\Delta + V)\psi(t)$

$\psi(t, x)$ wave function. $\hbar$: Planck const.

Quantum mechanics

Stationary Schrödinger eq.: $(\boldsymbol{\Delta}+\boldsymbol{V})\boldsymbol{\psi}=\mathbf{0}$

Harmonic function $\boldsymbol{h} = \boldsymbol{\psi}$.

Global coordinate $\boldsymbol{L}^2(|\boldsymbol{h}|^2\mathrm{d}\boldsymbol{x})$. Math

$(\boldsymbol{\Delta}+\boldsymbol{V})\boldsymbol{\psi}=\boldsymbol{E_m}\boldsymbol{\psi}$, $\boldsymbol{E_m}$: eigenvalue

New harmonic function $\boldsymbol{h_m} = \boldsymbol{\psi_m}$.

Local coordinate: $\boldsymbol{L}^2(|\boldsymbol{h_m}|^2\mathrm{d}\boldsymbol{x})$. Physics

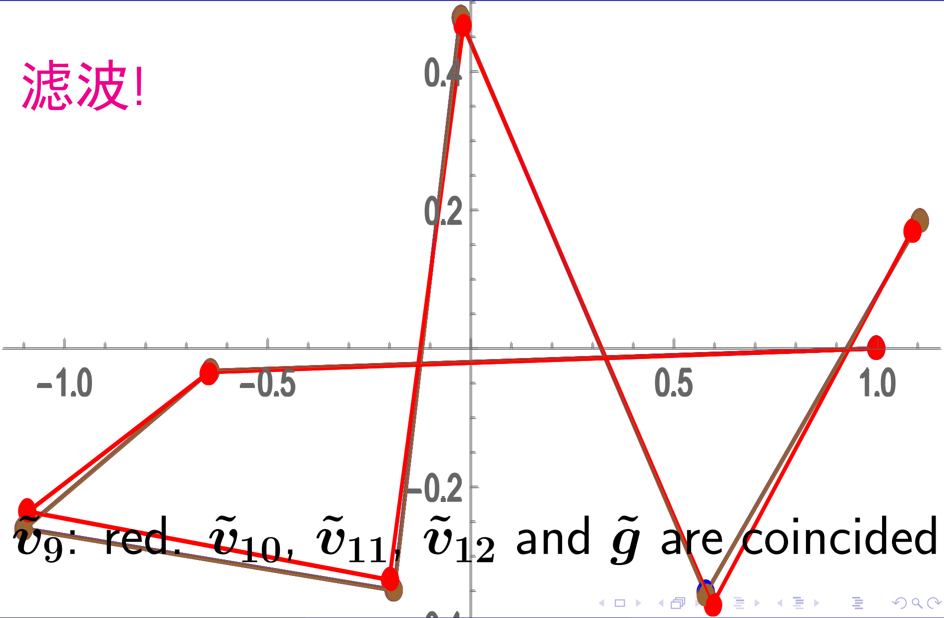
$\boldsymbol{E_m}$: unknown!

Einstein: "God does not play dice."

Complex stochastic processes

- SLE theory. One-dim complex BM, Lévy. "Conformally invariant processes in the plane". G.F. Lawler$'^{05}$
- Dirichlet form theory. M. Fukushima and M. Okada$'^{87}$. "On Dirichlet forms for plurisubharmonic functions". 5 papers
- Probability amplitude theory. "Complex Markov chains". V.P. Maslov$'^{70}$

滤波!



$\tilde{v}_9$: red. $\tilde{v}_{10}$, $\tilde{v}_{11}$, $\tilde{v}_{12}$ and $\tilde{g}$ are coincided

2. Computing the maximal eigenpair. Trid

$$E = \{k \in \mathbb{Z}_+ : 0 \leqslant k < N+1\}.$$

$$Q = \begin{pmatrix} -3 & 2 & & & 0 \\ 1 & -3 & 2 & & \\ & 1 & -3 & 2 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & 1 & -3 \end{pmatrix},$$

$$\lambda_0 + 3 = 2\sqrt{2}\cos\frac{2\pi}{N+2}$$

$$g_0(j) = 2^{-(j+1)/2}\sin\frac{(j+1)\pi}{N+2}, \qquad j \in E.$$

Shifted Inverse Algorithm

Given initial $(v_0, z_0) \approx (g_0, \lambda_0(-Q))$, suppose for $k \geqslant 1$, we have (v_{k-1}, z_{k-1}). Let w_k solve the equation

$$\big(-Q - z_{k-1}I\big)w_k = v_{k-1}.$$

Next, let $v_k = w_k/\|w_k\|, \qquad z_k = \delta_k^{-1}$

Then $(v_k, z_k) \to \big(g_0, \lambda_0(-Q)\big)$, $k \to \infty$.

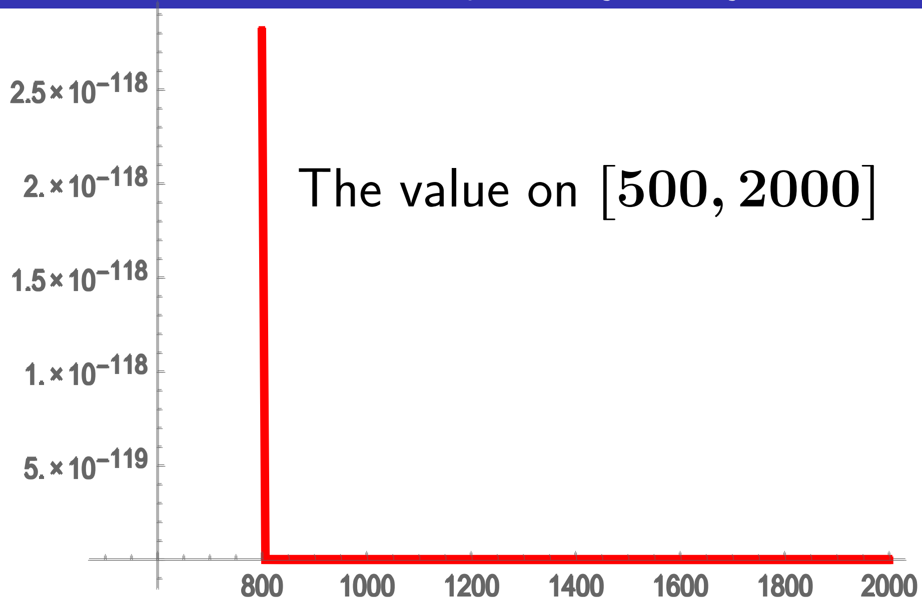
Isospectral matrix having conservative rows

$$E = \{k \in \mathbb{Z}_+ : 0 \leqslant k < N + 1\}.$$

$$\tilde{Q} = \begin{pmatrix} -3 & \frac{2^2-1}{2-1} & & & 0 \\ \frac{2^2-2}{2^2-1} & -3 & \frac{2^3-1}{2^2-1} & & \\ & \frac{2^3-2}{2^3-1} & -3 & \frac{2^4-1}{2^3-1} & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \frac{2^{N+1}-2}{2^{N+1}-1} & -3 \end{pmatrix},$$

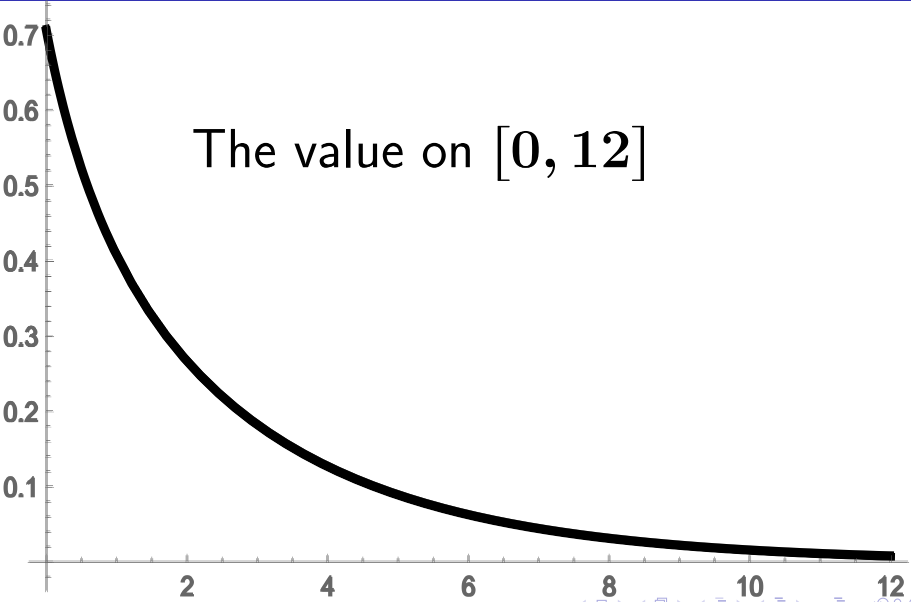
$$\lambda_0 + 3 = 2\sqrt{2}\cos\frac{2\pi}{N+2} \approx 2.82843, \ N \geqslant 2562.$$

The initial vector for $\tilde{Q}$ decays very fast

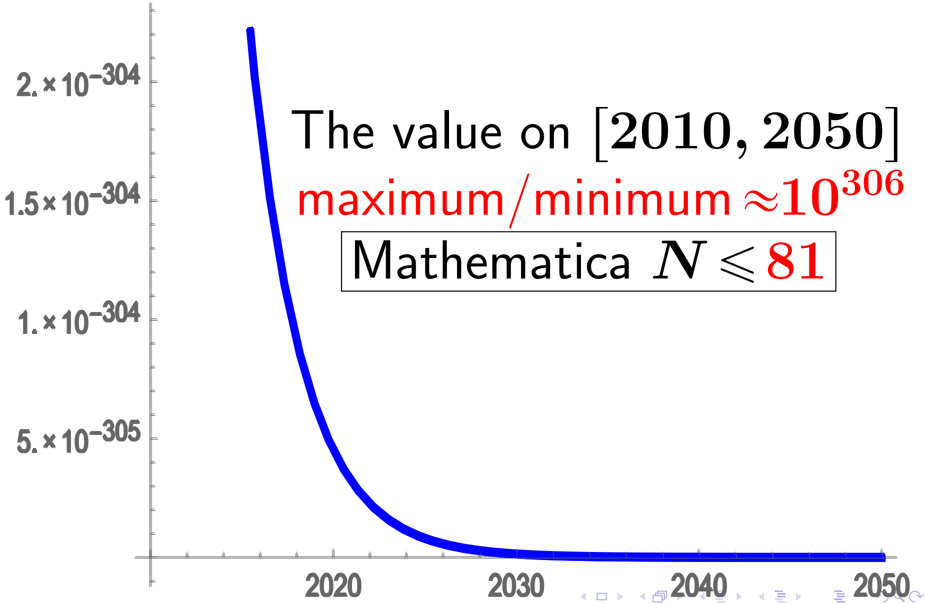


The value on $[500, 2000]$

The initial vector for $\widetilde{Q}$ decays very fast $\mathbf{0.7}\downarrow$



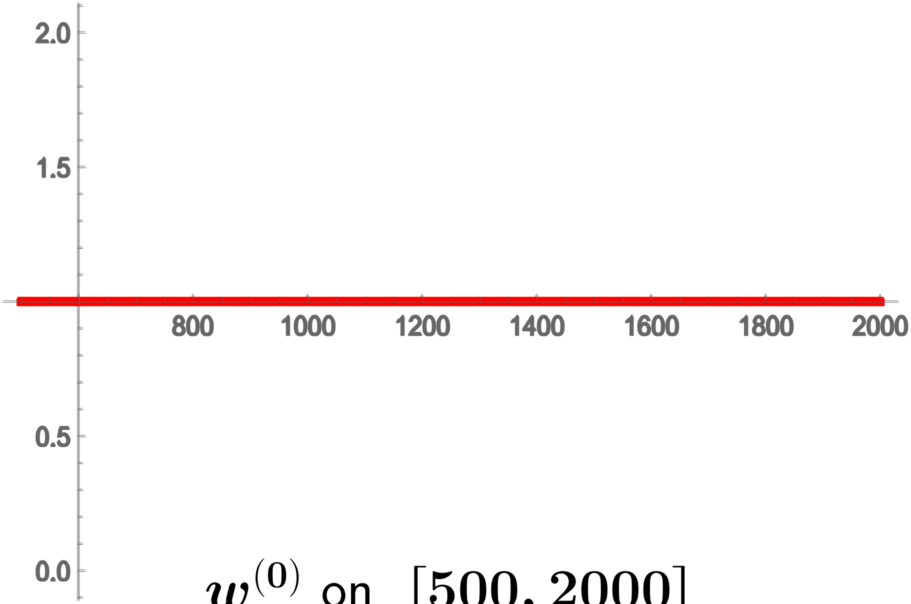
The initial vector for $\tilde{Q}$ decays very fast **0.7** $\downarrow$



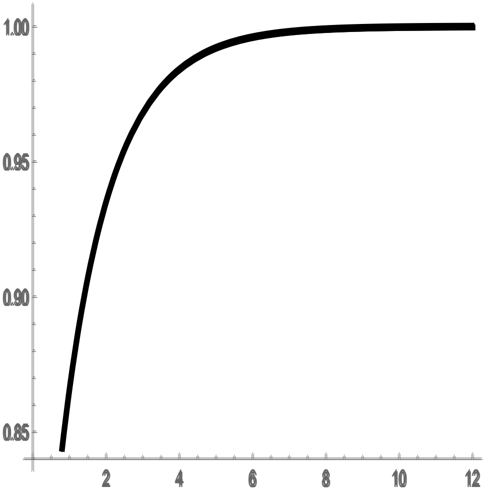
Q^{Sym}: symmetrization of $\widetilde{Q}$

$$Q^{\mathsf{Sym}} = \begin{pmatrix} -\mathbf{3} & \color{red}{\sqrt{\mathbf{2}}} & & & 0 \\ \color{red}{\sqrt{\mathbf{2}}} & -\mathbf{3} & \sqrt{\mathbf{2}} & & \\ & \sqrt{\mathbf{2}} & -\mathbf{3} & \sqrt{\mathbf{2}} & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \sqrt{\mathbf{2}} & -\mathbf{3} \end{pmatrix},$$

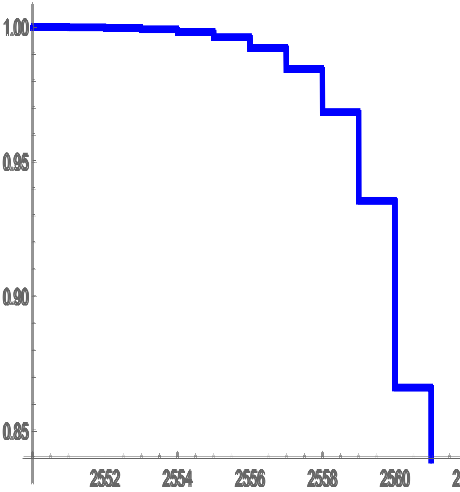
For the symmetrized matrix, sum of each row $\neq \mathbf{0}$, loss of martial arts. Isospectral transform $\Rightarrow$ non-symmetry. Unsolvable circulation. 处于绝境. 奇思妙想, 绝地逢生



$$w^{(0)} \text{ on } [500, 2000]$$



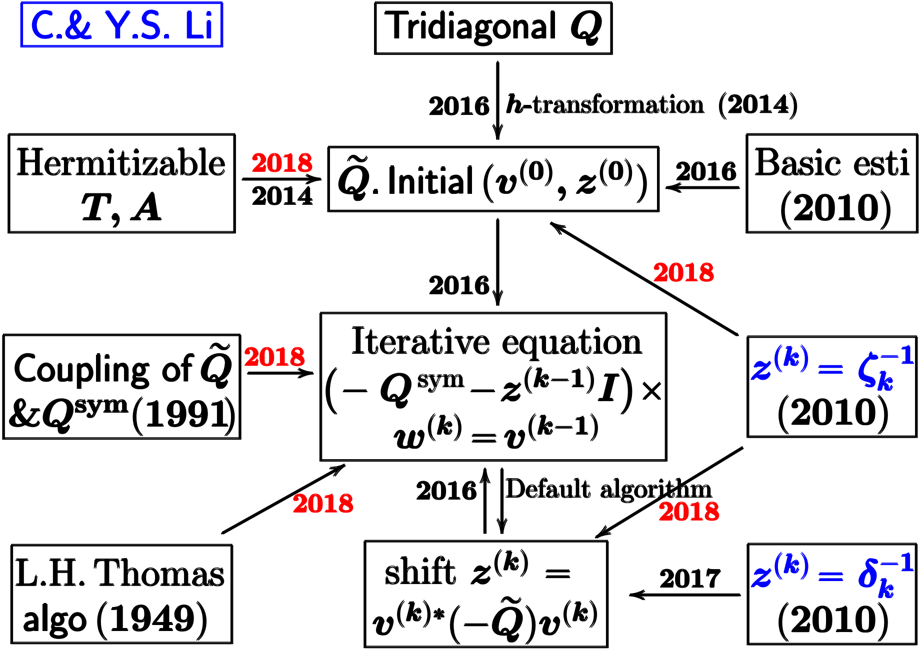
$w^{(0)}$ on $[0, 12]$



$w^{(0)}$ on $[2550, 2562]$

For our new algorithm,
maximum/minimum $= \sqrt{2} \ll 10^{306}$.

$N+1$	$z^{(0)}$	$z^{(1)}$	$z^{(2)}$	$z^{(3)}$
10^2	.171573	.172686	.172934	.172941
5×10^2	.171573	.171628	.171618	
10^3	.171573	.171584	.171587	
5×10^3	.171573	.171573	(1.597s)	
10^4	.171573	.171573	(6.578s)	
1.5×10^4	.171573	.171573	(29.16s)	


C.& Y.S. Li
Tridiagonal Q
2016
h-transformation (2014)
Hermitizable T, A
2018
2014
$\tilde{Q}$. Initial $(v^{(0)}, z^{(0)})$
2016
Basic esti (2010)
2016
2018
Coupling of $\tilde{Q}$ & Q^{sym} (1991)
2018
Iterative equation
$\left(-Q^{\text{sym}} - z^{(k-1)} I\right) \times w^{(k)} = v^{(k-1)}$
$z^{(k)} = \zeta_k^{-1}$ (2010)
2018
2016
Default algorithm
2018
L.H. Thomas algo (1949)
shift $z^{(k)} = v^{(k)*}(-\tilde{Q})v^{(k)}$
2017
$z^{(k)} = \delta_k^{-1}$ (2010)

3. Improved global algorithm(2017)

Single death $Q=(q_{ij})_{i,j=1}^{N}$: $\boxed{\text{Y.S. Li}}$

$$Q=\begin{pmatrix} -\frac{7}{3^2} & \frac{2}{3^3} & \frac{2}{3^4} & \cdots & \frac{2}{3^N} & \frac{1}{3^N} \\ \frac{2}{3} & -\frac{7}{3^2} & \frac{2}{3^3} & \cdots & \frac{2}{3^{N-1}} & \frac{1}{3^{N-1}} \\ & \frac{2}{3} & -\frac{7}{3^2} & \cdots & \frac{2}{3^{N-2}} & \frac{1}{3^{N-2}} \\ & & \ddots & \ddots & \ddots & \vdots \\ & 0 & & \frac{2}{3} & -\frac{7}{3^2} & \frac{1}{3^2} \\ & & & & \frac{2}{3} & -\frac{2}{3} \end{pmatrix}$$

Not co-zero! $A=(a_{kj}):=(-Q)^{-1}$

General positive matrix

Example [Yu-Hui Zhang].

$$A = (a_{kj})_{N\times N}, \quad a_{kj} = \frac{2^{k\wedge j} - 1}{2^j} + \mathbb{1}_{\{j \leqslant k\}}.$$

Using Mathematica (version 11.3), Maximal eigenvector is computable iff $N \leqslant \mathbf{39}$.
Without term $\mathbb{1}_{\{j\leqslant k\}}$, Mathematica is available iff $N \leqslant \mathbf{11}$. MatLab: $N \leqslant \mathbf{45}$.

Computational technique. $a_{ij} \geqslant 0,\ i \neq j$

- Invariant/Harmonic meas μ $\boxed{\text{A:irreducible}}$

$$Q := A - \mathrm{DiagMatrix}[A\mathbb{1}] \Rightarrow Q\mathbb{1} = \mathbb{0}.$$

Solve $\mu Q = 0$, $\mu_0 = 1$. $\boxed{\text{unique}}$

- Quasi-symmetrizing:

$$\hat{A} := \mathrm{Diag}(\mu^{1/2})A\mathrm{Diag}(\mu^{-1/2}).$$

- A symmetrizable w.r.t μ iff $\hat{A}$ symmetric: $\hat{A}^* - \hat{A} = 0$? $\boxed{\text{Diagonals, summing}}$

Quasi-symmetrization.

Quasi-symmetrizing:

$$\widehat{A} := \text{Diag}(\mu^{1/2})\, A\, \text{Diag}(\mu^{-1/2}).$$

Example (Continued)

$$A = (a_{kj}): a_{kj} = \frac{2^{k\wedge j} - 1}{2^j} + \mathbb{1}_{\{j \leqslant k\}},$$

without $\mathbb{1}_{\{j\leqslant k\}}$: symm wrt $\mu_k = 2^{-k}$;

with $\mathbb{1}_{\{j\leqslant k\}}$: q-symm wrt last μ_k, $N \leqslant 523$,

new q-symm $\mu_k = \left(2 \times 10^{2/39}\right)^{-k}$.

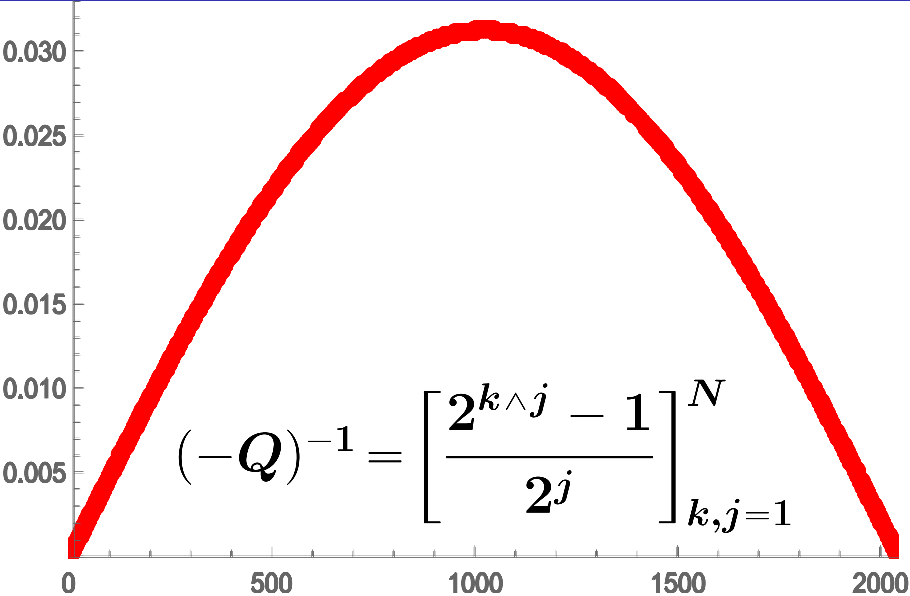
Results of quasi-symmetrization

$\mu_k = \left(2 \times 10^{2/39}\right)^{-k}$. $1700 \gg 39$

N	523	800	1000	1500	1700
$z^{(1)}$	**8.96625** **8.99793**	**8.97632** **8.99911**	**8.98126** **8.99943**	**8.98993** **8.99975**	**8.99111** **8.9998**
$z^{(2)}$	**8.99734** **8.99746**	**8.99885** **8.99891**	**8.99926** **8.9993**	**8.99967** **8.99969**	**8.99974** **8.99976**
$z^{(3)}$	**8.99744** **8.99744**	**8.9989** **8.9989**	**8.9993** **8.9993**	**8.99969** **8.99969**	**8.99976** **8.99976**

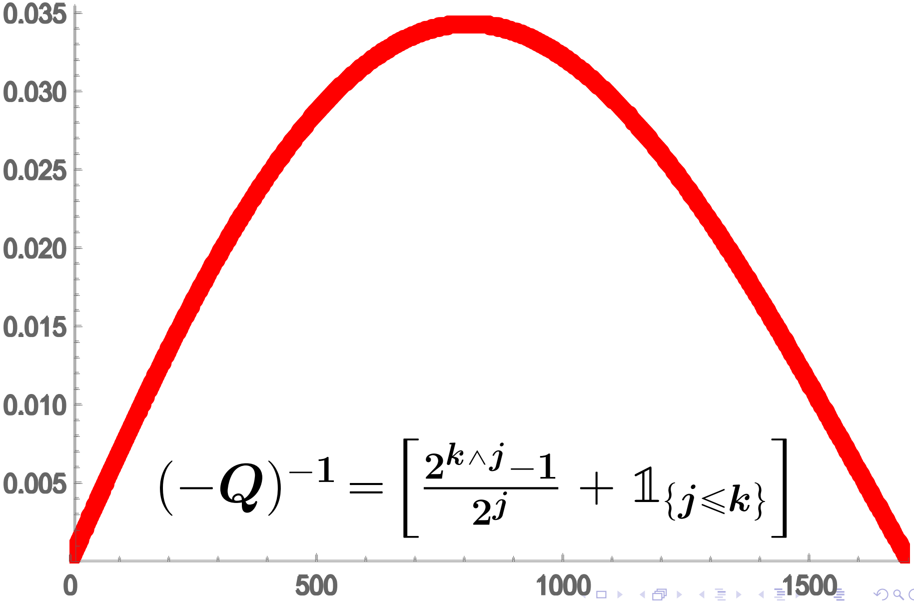
Here $z^{(k)}$ denotes the output of Step k of the lower/upper estimates of $\rho(A)$. Mathematica

$Q \sim (2, -3, 1),\ c_N = 2. \quad N = 2043$



$$(-Q)^{-1} = \left[\frac{2^{k\wedge j} - 1}{2^j}\right]_{k,j=1}^{N}$$

Q: single death. $N = 1700$



$$(-Q)^{-1} = \left[\frac{2^{k\wedge j}-1}{2^j} + \mathbb{1}_{\{j\leqslant k\}}\right]$$

随机测试. $A = (a_{ij}),\ a_{ij} \in (0, 10)$

使用普通笔记本电脑, MatLab.

- $N = 5000$. 7 小时, 2,326 个, 平均 11 秒/个. 一步迭代. 无错.
- $N = 1000$. 2 小时, 36,448 个, 平均 0.2 秒/个. 一步迭代. 无错.

$A = (-Q)^{-1} > 0$

Lemma

Let Q: a Q-matrix ($Q\mathbb{1} \leqslant 0$) on a countable E, derived by $\{P_t\}_{t\geqslant 0}$. Then, $(-Q)^{-1}$ $\exists$ & $\geqslant 0$ (pointwise, may be ∞). If moreover, Q is irreducible and $\{P_t\}_{t\geqslant 0}$ is transient, then $(-Q)^{-1} \in (0, \infty)$.

Proof As an increasing limit of $\boxed{\text{weak}}$

$$(\lambda I - Q)^{-1} = \int_0^\infty e^{-\lambda t} P_t \, \mathrm{d}t, \quad \lambda \downarrow 0.$$

华罗庚科普著作选集, 第二部分末文

- 研究要攻得进去, 还要打得出来.
- 鉴别一个学问家, 要看广度和深度.
  单是深, 可成为不错的专家, 但对整个科学的发展不足道.
  单是广, 可欺外行, 难有实质性成就.
- 学科之间实际上无不可逾越的鸿沟.
- 做学问需要有绅士风度, 新思想, 打个洞. 根据地, "漫"而非跳.

http://math0.bnu.edu.cn/~chenmf

The end!

Thank you, everybody!

谢谢大家!

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